SECTION 7.1 EXERCISES

Review Questions

1. On which derivative rule is integration by parts based?
2. How would you choose the term dv when evaluating $\int x^n e^{ax}\,dx$ using integration by parts?
3. How would you choose the term u when evaluting $\int x^n \cos ax\,dx$ using integration by parts?
4. Explain how integration by parts is used to evaluate a definite integral.
5. For what type of integrand is integration by parts useful?
6. How would you choose u and dv to simplify $\int x^4 e^{-2x}\,dx$?

Basic Skills

7–22. Integration by parts *Evaluate the following integrals.*

7. $\int x\cos x\,dx$
8. $\int x\sin 2x\,dx$
9. $\int te^t\,dt$
10. $\int 2xe^{3x}\,dx$
11. $\int x^2\sin 2x\,dx$
12. $\int se^{-2s}\,ds$
13. $\int x^2e^{4x}\,dx$
14. $\int \theta\sec^2\theta\,d\theta$
15. $\int x^2\ln x\,dx$
16. $\int x\ln x\,dx$
17. $\int \frac{\ln x}{x^{10}}\,dx$
18. $\int \sin^{-1}x\,dx$
19. $\int \tan^{-1}x\,dx$
20. $\int x\sec^{-1}x\,dx,\ x\ge 1$
21. $\int x\sin x\cos x\,dx$
22. $\int x\tan^{-1}(x^2)\,dx$

23–28. Repeated integration by parts *Evaluate the following integrals.*

23. $\int e^x\cos x\,dx$
24. $\int e^{3x}\cos 2x\,dx$
25. $\int e^{-x}\sin 4x\,dx$
26. $\int x^2\ln^2 x\,dx$
27. $\int t^3e^{-t}\,dt$
28. $\int e^{-2\theta}\sin 6\theta\,d\theta$

29–36. Definite integrals *Evaluate the following definite integrals.*

29. $\int_0^{\pi} x\sin x\,dx$
30. $\int_1^{e} \ln 2x\,dx$
31. $\int_0^{\pi/2} x\cos 2x\,dx$
32. $\int_0^{\ln 2} xe^x\,dx$
33. $\int_1^{e^2} x^2\ln x\,dx$
34. $\int_0^{1/\sqrt{2}} y\tan^{-1}y^2\,dy$
35. $\int_{1/2}^{\sqrt{3}/2} \sin^{-1}y\,dy$
36. $\int_{2/\sqrt{3}}^{2} z\sec^{-1}z\,dz$

37–40. Volumes of solids *Find the volume of the solid that is generated when the region is revolved as described.*

37. The region bounded by $f(x) = e^{-x}$, $x = \ln 2$, and the coordinate axes is revolved about the y-axis.
38. The region bounded by $f(x) = \sin x$ and the x-axis on $[0, \pi]$ is revolved about the y-axis.
39. The region bounded by $f(x) = x\ln x$ and the x-axis on $[1, e^2]$ is revolved about the x-axis.
40. The region bounded by $f(x) = e^{-x}$ and the x-axis on $[0, \ln 2]$ is revolved about the line $x = \ln 2$.

Further Explorations

41. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.
    a. $\int uv'\,dx = \left(\int u\,dx\right)\left(\int v'\,dx\right)$
    b. $\int uv'\,dx = uv - \int vu'\,dx$

42–45. Reduction formulas *Use integration by parts to derive the following reduction formulas.*

42. $\int x^n e^{ax}\,dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx$ for $a \ne 0$
43. $\int x^n\cos ax\,dx = \frac{x^n\sin ax}{a} - \frac{n}{a}\int x^{n-1}\sin ax\,dx$ for $a \ne 0$
44. $\int x^n\sin ax\,dx = -\frac{x^n\cos ax}{a} + \frac{n}{a}\int x^{n-1}\cos ax\,dx$ for $a \ne 0$
45. $\int \ln^n x\,dx = x\ln^n x - n\int \ln^{n-1}x\,dx$

46–49. Applying reduction formulas *Use the reduction formulas in Exercises 42–45 to evaluate the following integrals.*

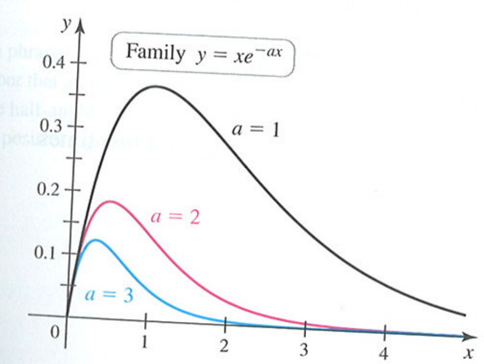
46. $\int x^2e^{3x}\,dx$
47. $\int x^2\cos 5x\,dx$
48. $\int x^3\sin x\,dx$
49. $\int \ln^4 x\,dx$

50–51. Integrals involving $\int \ln x\,dx$ *Use a substitution to reduce the following integrals to $\int \ln u\,du$. Then, evaluate the resulting integral.*

50. $\int \cos x\ln(\sin x)\,dx$
51. $\int \sec^2 x\ln(\tan x + 2)\,dx$

52. **Two methods**
    a. Evaluate $\int x\ln x^2\,dx$ using the substitution $u = x^2$ and evaluating $\int \ln u\,du$.
    b. Evaluate $\int x\ln x^2\,dx$ using integration by parts.
    c. Verify that your answers to parts (a) and (b) are consistent.
53. **Logarithm base b** Prove that $\int \log_b x\,dx = \frac{1}{\ln b}(x\ln x - x) + C$.

54. **Two integration methods** Evaluate $\int \sin x\cos x\,dx$ using integration by parts. Then evaluate the integral using a substitution. Reconcile your answers.
55. **Combining two integration methods** Evaluate $\int \cos(\sqrt{x})\,dx$ using a substitution followed by integration by parts.
56. **Combining two integration methods** Evaluate $\int_0^{\pi^2/4}\sin(\sqrt{x})\,dx$ using a substitution followed by integration by parts.
57. **Function defined as an integral** Find the arc length of the function $f(x) = \int_e^x \sqrt{\ln^2 t - 1}\,dt$ on $[e, e^3]$.
58. **A family of exponentials** The curves $y = xe^{-ax}$ are shown in the figure for $a = 1, 2,$ and 3.
    a. Find the area of the region bounded by $y = xe^{-x}$ and the x-axis on the interval $[0, 4]$.
    b. Find the area of the region bounded by $y = xe^{-ax}$ and the x-axis on the interval $[0, 4]$, where $a > 0$.
    c. Find the area of the region bounded by $y = xe^{-ax}$ and the x-axis on the interval $[0, b]$. Because this area depends on a and b, we call it $A(a, b)$, where $a > 0$ and $b > 0$.
    d. Use part (c) to show that $A(1, \ln b) = 4A(2, (\ln b)/2)$.
    e. Does this pattern continue? Is it true that $A(1, \ln b) = a^2A(a, (\ln b)/a)$?



59. **Solid of revolution** Find the volume of the solid generated when the region bounded by $y = \cos x$ and the x-axis on the interval $[0, \pi/2]$ is revolved about the y-axis.
60. **Between the sine and inverse sine** Find the area of the region bounded by the curves $y = \sin x$ and $y = \sin^{-1}x$ on the interval $[0, 1]$.
61. **Comparing volumes** Let R be the region bounded by $y = \sin x$ and the x-axis on the interval $[0, \pi]$. Which is greater, the volume of the solid generated when R is revolved about the x-axis or the volume of the solid generated when R is revolved about the y-axis?
62. **Log integrals** Use integration by parts to show that for $m \ne -1$,
$$\int x^m\ln x\,dx = \frac{x^{m+1}}{m+1}\left(\ln x - \frac{1}{m+1}\right) + C$$
and for $m = -1$,
$$\int \frac{\ln x}{x}\,dx = \frac{1}{2}\ln^2 x + C.$$

63. **A useful integral**
    a. Use integration by parts to show that if f' is continuous
$$\int xf'(x)\,dx = xf(x) - \int f(x)\,dx.$$
    b. Use part (a) to evaluate $\int xe^{3x}\,dx$.
64. **Integrating inverse functions**
    a. Let $y = f^{-1}(x)$, which means $x = f(y)$ and $dx = f'(y)\,dy$. Show that
$$\int f^{-1}(x)\,dx = \int yf'(y)\,dy.$$
    b. Use the result of Exercise 63 to show that
$$\int f^{-1}(x)\,dx = yf(y) - \int f(y)\,dy.$$
    c. Use the result of part (b) to evaluate $\int \ln x\,dx$ (express the result in terms of x).
    d. Use the result of part (b) to evaluate $\int \sin^{-1}x\,dx$.
    e. Use the result of part (b) to evaluate $\int \tan^{-1}x\,dx$.
65. **Integral of $\sec^3 x$** Use integration by parts to show that
$$\int \sec^3 x\,dx = \frac{1}{2}\sec x\tan x + \frac{1}{2}\int \sec x\,dx.$$
66. **Two useful exponential integrals** Use integration by parts to derive the following formulas for real numbers a and b.
$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2 + b^2} + C$$
$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2 + b^2} + C$$

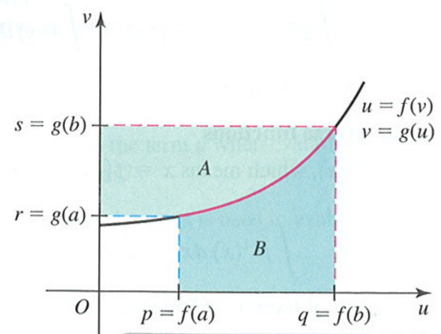
Applications

T 67. **Oscillator displacements** Suppose an oscillator (such as a pendulum or a mass on a spring) that is slowed by friction has the position function $s(t) = e^{-t}\sin t$.
    a. Graph the position function. At what times does the oscillator pass through the position $s = 0$?
    b. Find the average value of the position on the interval $[0, \pi]$.
    c. Generalize part (b) and find the average value of the position on the interval $[n\pi, (n + 1)\pi]$ for $n = 0, 1, 2, \ldots$.
    d. Let a_n be the absolute value of the average position on the intervals $[n\pi, (n + 1)\pi]$ for $n = 0, 1, 2, \ldots$. Describe the pattern in the numbers $a_0, a_1, a_2, \ldots$.

Additional Exercises

68. **Find the error** Suppose you evaluate $\int \frac{dx}{x}$ using integration by parts. With $u = 1/x$ and $dv = dx$, you find that $du = -1/x^2\,dx$, $v = x$, and
$$\int \frac{dx}{x} = \left(\frac{1}{x}\right)x - \int x\left(-\frac{1}{x^2}\right)dx = 1 + \int \frac{dx}{x}.$$
You conclude that $0 = 1$. Explain the problem with the calculation.

69. Proof without words Explain how the diagram in the figure illustrates integration by parts for definite integrals.



Area of A + Area of $B = qs - pr$

70. An identity Show that if f and g have continuous second derivatives and $f(0) = f(1) = g(0) = g(1) = 0$, then

$$\int_0^1 f''(x)g(x)\,dx = \int_0^1 f(x)g''(x)\,dx.$$

71. Possible and impossible integrals Let $I_n = \int x^n e^{-x^2}\,dx$, where n is a nonnegative integer.

- **a.** $I_0 = \int e^{-x^2}\,dx$ cannot be expressed in terms of elementary functions. Evaluate I_1.
- **b.** Use integration by parts to evaluate I_3.
- **c.** Use integration by parts and the result of part (b) to evaluate I_5.
- **d.** Show that, in general, if n is odd, then $I_n = -\frac{1}{2}e^{-x^2}p_{n-1}(x)$, where p_{n-1} is an even polynomial of degree $n - 1$.
- **e.** Argue that if n is even, then I_n cannot be expressed in terms of elementary functions.

72. Looking ahead (to Chapter 9) Suppose that a function f has derivatives of all orders near $x = 0$. By the Fundamental Theorem of Calculus,

$$f(x) - f(0) = \int_0^x f'(t)\,dt.$$

- **a.** Evaluate the integral using integration by parts to show that

$$f(x) = f(0) + xf'(0) + \int_0^x f''(t)(x - t)\,dt.$$

- **b.** Show (by observing a pattern or using induction) that by integrating by parts n times,

$$f(x) = f(0) + xf'(0) + \frac{1}{2!}x^2 f''(0) + \cdots + \frac{1}{n!}x^n f^{(n)}(0) + \frac{1}{n!}\int_0^x f^{(n+1)}(t)(x - t)^n\,dt + \cdots$$

This expression, called the *Taylor series* for f at $x = 0$, is revisited in Chapter 9.

QUICK CHECK ANSWERS

1. Let $u = x$ and $dv = \cos x\,dx$.
2. $\frac{d}{dx}(x \ln x - x + C) = \ln x$
3. Integration by parts must be applied five times. ◄

7.2 Trigonometric Integrals

At the moment, our inventory of integrals involving trigonometric functions is rather limited. For example, we can integrate $\sin ax$ and $\cos ax$, where a is a constant, but missing from the list are integrals of $\tan ax$, $\cot ax$, $\sec ax$, and $\csc ax$. It turns out that integrals of powers of trigonometric functions, such as $\int \cos^5 x\,dx$ and $\int \cos^2 x \sin^4 x\,dx$, are also important. The goal of this section is to develop techniques for integrating integrals involving trigonometric functions. These techniques are indispensable when we use *trigonometric substitutions* in the next section.

Integrating Powers of sin x and cos x

Two strategies are employed when evaluating integrals of the form $\int \sin^m x\,dx$ or $\int \cos^n x\,dx$, where m and n are positive integers. Both use trigonometric identities to recast the integrand, as shown in the first example.

EXAMPLE 1 Powers of sine or cosine Evaluate the following integrals.

a. $\int \cos^5 x\,dx$ **b.** $\int \sin^4 x\,dx$

SOLUTION

➤ Pythagorean identities:

$\cos^2 x + \sin^2 x = 1$

$1 + \tan^2 x = \sec^2 x$

$\cot^2 x + 1 = \csc^2 x$

a. Integrals involving odd powers of $\cos x$ (or $\sin x$) are most easily evaluated by splitting off a single factor of $\cos x$ (or $\sin x$). In this case, we rewrite $\cos^5 x$ as $\cos^4 x \cdot \cos x$. Now, $\cos^4 x$ can be written in terms of $\sin x$ using the identity $\cos^2 x = 1 - \sin^2 x$. The result is an integrand that readily yields to the substitution $u = \sin x$:

$$\begin{aligned}
\int \cos^5 x\,dx &= \int \cos^4 x \cdot \cos x\,dx && \text{Split off } \cos x.\\
&= \int (1 - \sin^2 x)^2 \cdot \cos x\,dx && \text{Pythagorean identity}\\
&= \int (1 - u^2)^2\,du && \text{Let } u = \sin x;\ du = \cos x\,dx.\\
&= \int (1 - 2u^2 + u^4)\,du && \text{Expand.}\\
&= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C && \text{Integrate.}\\
&= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C && \text{Replace } u \text{ with } \sin x.
\end{aligned}$$

b. With even powers of $\sin x$ or $\cos x$, we use the half-angle formulas

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

➤ Use the phrase "sine is minus" to remember that a minus sign is associated with the half-angle formula for $\sin^2 x$, while a positive sign is used for $\cos^2 x$.

to reduce the powers in the integrand:

$$\begin{aligned}
\int \sin^4 x\,dx &= \int \Big(\underbrace{\frac{1 - \cos 2x}{2}}_{\sin^2 x}\Big)^2 dx && \text{Half-angle formula}\\
&= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\,dx && \text{Expand the integrand.}
\end{aligned}$$

Using the half-angle formula again for $\cos^2 2x$, the evaluation may be completed:

$$\begin{aligned}
\int \sin^4 x\,dx &= \frac{1}{4}\int \Big(1 - 2\cos 2x + \underbrace{\frac{1 + \cos 4x}{2}}_{\cos^2 2x}\Big)dx && \text{Half-angle formula}\\
&= \frac{1}{4}\int \Big(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\Big)dx && \text{Simplify.}\\
&= \frac{3x}{8} - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C && \text{Evaluate the integrals.}
\end{aligned}$$

Related Exercises 9–12 ◄

QUICK CHECK 1 Evaluate $\int \sin^3 x\,dx$ by splitting off a factor of $\sin x$, rewriting $\sin^2 x$ in terms of $\cos x$, and using an appropriate u-substitution. ◄